Modular invariance of bosonic string on orbifolds

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Abstract

I construct a complete 1-loop partition function of a bosonic closed string on orbifolds. Furthermore, I derive sufficient conditions for the modular invariance of the partition function.

1 Introduction

Orbifold compactification models of the heterotic string [1] are candidates for unified theories. In the string theory, modular invariance is necessary for consistency. But modular invariance of orbifold models is non-trivial. A modular invariance condition of orbifold models is naively the level matching [2, 3]. In contrast, a free fermion model was built as another possibility [4]. Although the free fermion model is given by fermionizing all internal coordinates, the fermionization on orbifolds is restricted on the $\mathbf{Z}_2$ orbifold. Therefore we should consider the internal coordinates as bosonic variables so as to generalize the model to include general orbifolds. A 1-loop partition function of a bosonic closed string on orbifolds was constructed in the ref.[5]. But I think that this partition function is not complete in the points explained in the text. In this paper I construct a complete 1-loop partition function of a bosonic closed string on orbifolds. Furthermore, I derive sufficient conditions for the modular invariance of the partition function.

2 Preliminaries

In this section I set up the framework for this paper. I consider a heterotic string compactified to four space-time dimensions. In the light-cone gauge, there are the following world-sheet degrees of freedom: eight transverse bosons $X^i(z,\overline{z})$, eight right-moving transverse fermions $\tilde{\psi}^i(\overline{z})$ where $i = 2, 3, \ldots, 9$, and sixteen left-moving bosons $X_L^I(z)$ where $I = 10, 11, \ldots, 25$. Here the world-sheet coordinates are $z = e^{-i(\sigma^1+i\sigma^2)}$ and $\overline{z} = e^{i(\sigma^1-i\sigma^2)}$. In this paper I particularly pay attention to the bosons X^n where $n = 4, 5, \ldots, 9$ (corresponding to six internal coordinates).

An orbifold is obtained from flat space by the following identification under a discrete group G. An element g of G acts on the coordinates as a rotation θ and a translation $2\pi R$,

$$g : X^n \to \theta^{nm} X^m + 2\pi R^n, \tag{2.1}$$

where $m, n = 4, 5, \ldots, 9$. Let N be the smallest integer such as $\theta^N = 1$, then this orbifold is called a $\mathbf{Z}_N$ orbifold. It is convenient to change the basis of the coordinates so that the

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rotation θ becomes a diagonal matrix. Then the real bosons X^n become complex bosons ϕ^a where $a = 1, 2, 3$. In this basis, the action of g becomes

$$g : \phi^a \to e^{2\pi i \overline{\alpha v^a}} \phi^a + 2\pi \ell^a, \tag{2.2}$$

where α is an integer, v^a is a multiple of $1/N$ in the $\mathbf{Z}_N$ orbifold, and ℓ is an element of a 3-dimensional complex lattice Λ. The lattice Λ must be invariant under the rotation diag$(e^{2\pi i v^1}, e^{2\pi i v^2}, e^{2\pi i v^3})$ so that the orbifold is well-defined. $\overline{\alpha v^a}$ is defined by $\overline{\alpha v^a} = \alpha v^a \pmod 1$ and $0 \leq \overline{\alpha v^a} < 1$. Then the elements of the group G are in one-to-one correspondence with the parameters $\overline{\alpha v}$ and ℓ, which are denoted by $g(\overline{\alpha v}, \ell)$. A representation of $g(\overline{\alpha v}, \ell)$ is defined by

$$g^{-1}(\overline{\alpha v}, \ell) \phi^a g(\overline{\alpha v}, \ell) = e^{2\pi i \overline{\alpha v^a}} \phi^a + 2\pi \ell^a, \tag{2.3}$$

and $g(\overline{\alpha v}, \ell)$ is a unitary operator.

3 Partition function

In this section I construct the 1-loop partition function of the bosonic field ϕ on the orbifold. The heterotic string theory contains only the closed string. The 1-loop world-sheet of a closed string is a torus. The torus is described by the identifications $z \cong e^{2\pi i} z$ and $z \cong e^{2\pi i \tau} z$, where τ is a complex number. Therefore the field ϕ^a satisfies the periodic boundary conditions $\phi^a(e^{2\pi i} z, e^{-2\pi i} \overline{z}) = \phi^a(z, \overline{z})$ and $\phi^a(e^{2\pi i \tau} z, e^{-2\pi i \overline{\tau}} \overline{z}) = \phi^a(z, \overline{z})$. On the orbifold, because of the identification under the group G, the following boundary conditions are allowed,

$$\phi^a(e^{2\pi i} z, e^{-2\pi i} \overline{z}) = g^{-1}(\overline{\alpha v}, \ell) \phi^a(z, \overline{z}) g(\overline{\alpha v}, \ell), \tag{3.1a}$$

$$\phi^a(e^{2\pi i \tau} z, e^{-2\pi i \overline{\tau}} \overline{z}) = g^{-1}(\overline{\beta v}, \ell') \phi^a(z, \overline{z}) g(\overline{\beta v}, \ell'), \tag{3.1b}$$

where α and β are integers and $\ell, \ell' \in \Lambda$. Then the partition function is given by

$$Z(\tau) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{N-1} Z_{\overline{\beta v}}^{\overline{\alpha v}}(\tau), \tag{3.2}$$

with

$$Z_{\overline{\beta v}}^{\overline{\alpha v}}(\tau) = \sum_{\ell} \sum_{\ell'} \mathrm{Tr} \left\{ q^{H_{\overline{\alpha v}, \ell}} \overline{q}^{\tilde{H}_{\overline{\alpha v}, \ell}} g(\overline{\beta v}, \ell') \right\}, \tag{3.3}$$

where $q = e^{2\pi i \tau}$, and $H_{\overline{\alpha v}, \ell}$ and $\tilde{H}_{\overline{\alpha v}, \ell}$ are left- and right-moving Hamiltonians respectively.

Let $\phi^a_{\overline{\alpha v}, \ell}$ be a field which satisfies the boundary condition (3.1a). Then I obtain

$$\phi^a_{\overline{\alpha v}, \ell}(e^{2\pi i} z, e^{-2\pi i} \overline{z}) = e^{2\pi i \overline{\alpha v^a}} \phi^a_{\overline{\alpha v}, \ell}(z, \overline{z}) + 2\pi \ell^a. \tag{3.4}$$

In case of $\overline{\alpha v^a} = 0$, this field has the mode expansion

$$\phi^a_{\overline{\alpha v}, \ell}(z, \overline{z}) = \chi^a - i\rho^a_L \log z - i\rho^a_R \log \overline{z} + i \sum_{n=1}^{\infty} \frac{1}{n} (\beta^a_n z^{-n} + \gamma^{\dagger a}_n z^n + \tilde{\beta}^a_n \overline{z}^{-n} + \tilde{\gamma}^{\dagger a}_n \overline{z}^n), \tag{3.5}$$

where I defined the left- and right-moving momenta respectively by

$$\rho^a_L = \rho^a + \frac{\ell^a}{2}, \quad \rho^a_R = \rho^a - \frac{\ell^a}{2}, \tag{3.6}$$

χ is the center of mass position, $\gamma_n^\dagger$ and $\tilde{\gamma}_n^\dagger$ are the creation operators, and β_n and $\tilde{\beta}_n$ are the annihilation operators. The commutation relations of these operators are

$$[\chi^a, \rho^{\dagger b}] = [\chi_L^a, \rho_L^{\dagger b}] = [\chi_R^a, \rho_R^{\dagger b}] = i\delta^{ab}, \tag{3.7a}$$

$$[\beta_m^a, \beta_n^{\dagger b}] = [\gamma_m^a, \gamma_n^{\dagger b}] = [\tilde{\beta}_m^a, \tilde{\beta}_n^{\dagger b}] = [\tilde{\gamma}_m^a, \tilde{\gamma}_n^{\dagger b}] = m\delta_{mn}\delta^{ab}, \tag{3.7b}$$

where χ_L^a and χ_R^a are the left- and right-moving parts of χ^a, respectively. In case of $\overline{\alpha v^a} \neq 0$, on the other hand, the field $\phi^a_{\overline{\alpha v},\ell}$ has the mode expansion

$$\begin{aligned} \phi^a_{\overline{\alpha v},\ell}(z,\overline{z}) = \chi^a + i\sum_{n=1}^{\infty}\Bigg(& \frac{\beta^a_{n-\overline{\alpha v}}}{n-\overline{\alpha v^a}} z^{-n+\overline{\alpha v^a}} + \frac{\gamma^{\dagger a}_{n+\overline{\alpha v}-1}}{n+\overline{\alpha v^a}-1} z^{n+\overline{\alpha v^a}-1} \\ & + \frac{\tilde{\beta}^a_{n+\overline{\alpha v}-1}}{n+\overline{\alpha v^a}-1}\overline{z}^{-n-\overline{\alpha v^a}+1} + \frac{\tilde{\gamma}^{\dagger a}_{n-\overline{\alpha v}}}{n-\overline{\alpha v^a}}\overline{z}^{n-\overline{\alpha v^a}}\Bigg). \end{aligned} \tag{3.8}$$

The commutation relations of the creation and annihilation operators are similar to (3.7b). Then, to satisfy the boundary condition (3.4), χ^a must be a fixed point which satisfies

$$e^{2\pi i\overline{\alpha v^a}}\chi^a + 2\pi\ell^a = \chi^a. \tag{3.9}$$

Now I decompose the left-moving Hamiltonian $H_{\overline{\alpha v},\ell}$ into $H_{\rho;\,\overline{\alpha v},\ell}$ and $H_{N;\,\overline{\alpha v}}$, where $H_{\rho;\,\overline{\alpha v},\ell}$ depends on the momentum ρ_L. Let the right-moving Hamiltonian $\tilde{H}_{\overline{\alpha v},\ell}$ be similarly divided.

$$H_{\overline{\alpha v},\ell} = H_{\rho;\,\overline{\alpha v},\ell} + H_{N;\,\overline{\alpha v}}, \quad \tilde{H}_{\overline{\alpha v},\ell} = \tilde{H}_{\rho;\,\overline{\alpha v},\ell} + \tilde{H}_{N;\,\overline{\alpha v}}. \tag{3.10}$$

Here

$$H_{\rho;\,\overline{\alpha v},\ell} = \rho_L^\dagger \circ \rho_L = (\rho^\dagger + \overline{\ell}/2)\circ(\rho + \ell/2), \tag{3.11a}$$

$$\tilde{H}_{\rho;\,\overline{\alpha v},\ell} = \rho_R^\dagger \circ \rho_R = (\rho^\dagger - \overline{\ell}/2)\circ(\rho - \ell/2), \tag{3.11b}$$

(the symbol $\circ$ is defined by $x \circ y = \sum_{a=1}^{3}\delta_{\overline{\alpha v^a},0}x^a y^a$)

$$H_{N;\,\overline{\alpha v}} = \sum_{n=1}^{\infty}\left\{(n-\overline{\alpha v})\cdot N_{n-\overline{\alpha v}} + (n+\overline{\alpha v}-1)\cdot N'_{n+\overline{\alpha v}-1}\right\} + \frac{1}{2}\overline{\alpha v}\cdot(1-\overline{\alpha v}) - \frac{3}{12}, \tag{3.11c}$$

$$\tilde{H}_{N;\,\overline{\alpha v}} = \sum_{n=1}^{\infty}\left\{(n+\overline{\alpha v}-1)\cdot \tilde{N}_{n+\overline{\alpha v}-1} + (n-\overline{\alpha v})\cdot \tilde{N}'_{n-\overline{\alpha v}}\right\} + \frac{1}{2}\overline{\alpha v}\cdot(1-\overline{\alpha v}) - \frac{3}{12}. \tag{3.11d}$$

$N_r^a, N_r'^a, \tilde{N}_r^a$ and $\tilde{N}_r'^a$ are the occupation numbers defined as $N_r^a = \beta_r^{\dagger a}\beta_r^a/r^a$, $N_r'^a = \gamma_r^{\dagger a}\gamma_r^a/r^a$, $\tilde{N}_r^a = \tilde{\beta}_r^{\dagger a}\tilde{\beta}_r^a/r^a$ and $\tilde{N}_r'^a = \tilde{\gamma}_r^{\dagger a}\tilde{\gamma}_r^a/r^a$, but $N_0'^a = \tilde{N}_0^a = 0$.

Now an element $g_{\overline{\alpha v},\ell}(\overline{\beta v},\ell')$ of the group G is defined by

$$g^{-1}_{\overline{\alpha v},\ell}(\overline{\beta v},\ell')\phi^a_{\overline{\alpha v},\ell}(z,\overline{z})g_{\overline{\alpha v},\ell}(\overline{\beta v},\ell') = e^{2\pi i\overline{\beta v^a}}\phi^a_{\overline{\alpha v},\ell}(z,\overline{z}) + 2\pi\ell'^a. \tag{3.12}$$

up to an arbitrary constant $C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}$. This constant is a phase factor $\left(\left|C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}\right|^2 = 1\right)$, because $g_{\overline{\alpha v},\ell}(\overline{\beta v},\ell')$ is a unitary operator. In ref.[5], this constant was chosen to be independent of ℓ and ℓ'. But ℓ and ℓ' dependence of this constant is necessary for convergence of the partition function (3.3) summed over all ℓ and ℓ'. In this paper, therefore, I consider ℓ and ℓ' dependence

of this constant. Now I divide $g_{\overline{\alpha v},\ell}(\overline{\beta v},\ell')$ except this constant into a factor $h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell')$ which depends the momenta and the other factor $h_{N;\overline{\alpha v}}(\overline{\beta v})$,

$$g_{\overline{\alpha v},\ell}(\overline{\beta v},\ell') = C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell') h_{N;\overline{\alpha v}}(\overline{\beta v}), \tag{3.13}$$

where

$$h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell') = \exp\left\{-2\pi i(\overline{\ell'}\circ\rho + \ell'\circ\rho^\dagger)\right\}\exp\left\{2\pi i\overline{\beta v}\circ(K_L+K_R)\right\}, \tag{3.14a}$$

$$h_{N;\overline{\alpha v}}(\overline{\beta v}) = \exp\left\{2\pi i\overline{\beta v}\cdot(J_{\overline{\alpha v}}+\tilde{J}_{\overline{\alpha v}})\right\}, \tag{3.14b}$$

$$K^a_L = -i\left(\chi^a_L\rho^{\dagger a}_L - \rho^a_L\chi^{\dagger a}_L\right), \qquad K^a_R = -i\left(\chi^a_R\rho^{\dagger a}_R - \rho^a_R\chi^{\dagger a}_R\right), \tag{3.14c}$$

$$J^a_{\overline{\alpha v}} = \sum_{n=1}^{\infty}\left(N^a_{n-\overline{\alpha v}} - N'^a_{n+\overline{\alpha v}-1}\right), \quad \tilde{J}^a_{\overline{\alpha v}} = \sum_{n=1}^{\infty}\left(\tilde{N}^a_{n+\overline{\alpha v}-1} - \tilde{N}'^a_{n-\overline{\alpha v}}\right). \tag{3.14d}$$

In case of $\overline{\alpha v^a}\neq 0$, χ^a must be a fixed point which satisfies

$$e^{2\pi i\overline{\beta v^a}}\chi^a + 2\pi\ell'^a = \chi^a, \tag{3.15}$$

so that the field (3.8) is transformed according to (3.12), because there is no noncommutative operator to χ^a.

Thus, the partition function (3.3) is given by product of two functions $Z_\rho{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ and $Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$,

$$Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) = Z_\rho{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau), \tag{3.16}$$

with

$$Z_\rho{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) = \sum_\ell\sum_{\ell'} C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}\mathrm{Tr}\left\{q^{H_{\rho;\overline{\alpha v},\ell}}\overline{q}^{\tilde{H}_{\rho;\overline{\alpha v},\ell}} h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell')\right\}, \tag{3.17a}$$

$$Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) = \mathrm{Tr}\left\{q^{H_{N\,\overline{\alpha v}}}\overline{q}^{\tilde{H}_{N\,\overline{\alpha v}}} h_{N\,\overline{\alpha v}}(\overline{\beta v})\right\}. \tag{3.17b}$$

First, I calculate the function $Z_\rho{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. I consider common eigenstates of the operators $H_{\rho;\overline{\alpha v},\ell}$, $\tilde{H}_{\rho;\overline{\alpha v},\ell}$ and $h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell')$ so as to calculate the function $Z_\rho{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. In ref.[5], these eigenstates are not discussed. But, if these eigenstates are not discussed then the function $Z_\rho{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ cannot be exactly derived. In this paper, therefore, I derive these eigenstates and their eigenvalues. Let $|\rho_L,\rho_R\rangle_{\overline{\alpha v}}$ be a common eigenstate of both operators $\hat{\rho}_L$ and $\hat{\rho}_R$. This state is an eigenstate of both $H_{\rho;\overline{\alpha v},\ell}$ and $\tilde{H}_{\rho;\overline{\alpha v},\ell}$. But this state is not an eigenstate of $h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell')$, because the operator $e^{2\pi i\overline{\beta v}\circ(K_L+K_R)}$ in $h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell')$ rotate the momenta ρ_L and ρ_R of this state,

$$\hat{\rho}^a_L\left(e^{2\pi i\overline{\beta v}\circ(K_L+K_R)}|\rho_L,\rho_R\rangle_{\overline{\alpha v}}\right) = e^{2\pi i\overline{\beta v^a}}\rho^a_L\left(e^{2\pi i\overline{\beta v}\circ(K_L+K_R)}|\rho_L,\rho_R\rangle_{\overline{\alpha v}}\right), \tag{3.18a}$$

$$\hat{\rho}^a_R\left(e^{2\pi i\overline{\beta v}\circ(K_L+K_R)}|\rho_L,\rho_R\rangle_{\overline{\alpha v}}\right) = e^{2\pi i\overline{\beta v^a}}\rho^a_R\left(e^{2\pi i\overline{\beta v}\circ(K_L+K_R)}|\rho_L,\rho_R\rangle_{\overline{\alpha v}}\right). \tag{3.18b}$$

Therefore I consider linear combinations of the states,

$$\left|\overline{\rho_L}\circ\rho_L,\overline{\rho_R}\circ\rho_R,k\right\rangle^{\overline{\alpha v}}_{\overline{\beta v},\ell'} = \frac{1}{\sqrt{m}}\sum_{n=0}^{m-1}\exp\left(2\pi i\lambda^k_n\right)\exp\left\{2\pi i n\overline{\beta v}\circ(K_L+K_R)\right\}|\rho_L,\rho_R\rangle_{\overline{\alpha v}}, \tag{3.19}$$

where m is the smallest positive integer such that $\overline{m\beta v^a} = 0$ for $\forall a$ ($\overline{\alpha v^a} = 0$, ($\rho_L^a \neq 0$ or $\rho_R^a \neq 0$)), and $k = 0, 1, \ldots, m-1$. Let λ_0^k be zero. In addition, let λ_n^k satisfy

$$\frac{1}{m} \sum_{n=0}^{m-1} \exp\left(-2\pi i \lambda_n^k\right) \exp\left(2\pi i \lambda_n^{k'}\right) = \delta^{kk'}, \tag{3.20}$$

so that these states (3.19) belong to the orthonormal system. These states are eigenstates of both $H_{\rho;\overline{\alpha v},\ell} = \rho_L^\dagger \circ \rho_L$ and $\tilde{H}_{\rho;\overline{\alpha v},\ell} = \rho_R^\dagger \circ \rho_R$ because of (3.18a) and (3.18b). The operation of $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$ to these states becomes

$$\begin{aligned}
&h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell') \left|\overline{\rho_L} \circ \rho_L, \overline{\rho_R} \circ \rho_R, k\right\rangle^{\overline{\alpha v}}_{\overline{\beta v},\ell'} \\
&= \frac{1}{\sqrt{m}} \sum_{n=0}^{m-1} \exp\left(2\pi i \lambda_n^k\right) \exp\left\{-2\pi i (\overline{\ell'} \circ \rho + \ell' \circ \rho^\dagger)\right\} \exp\left\{2\pi i \overline{\beta v} \circ (K_L + K_R)\right\} \\
&\qquad \times \exp\left\{2\pi i \overline{n\beta v} \circ (K_L + K_R)\right\} \left|\rho_L, \rho_R\right\rangle_{\overline{\alpha v}} \qquad\qquad (3.21) \\
&= \frac{1}{\sqrt{m}} \sum_{n=0}^{m-1} \exp\left(2\pi i \lambda_n^k\right) \exp\left\{-2\pi i \left(\overline{\ell'} \circ e^{2\pi i \overline{(n+1)\beta v}} \rho + \text{c.c.}\right)\right\} \\
&\qquad \times \exp\left\{2\pi i \overline{(n+1)\beta v} \circ (K_L + K_R)\right\} \left|\rho_L, \rho_R\right\rangle_{\overline{\alpha v}},
\end{aligned}$$

where λ_n^k are

$$\lambda_n^k = n\lambda_k' - \left\{\overline{\ell'} \circ \left(e^{2\pi i \overline{\beta v}} + e^{2\pi i \overline{2\beta v}} + \cdots + e^{2\pi i \overline{n\beta v}}\right) \rho + \text{c.c.}\right\} \mod 1, \tag{3.22}$$

so that these states are eigenstates of $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$. Then the eigenvalue of $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$ is $\exp(-2\pi i \lambda_k')$. If $\rho^a \neq 0$ then $\rho_L^a \neq 0$ or $\rho_R^a \neq 0$ because of the relations (3.6). Therefore, if $\rho^a \neq 0$ then $\overline{m\beta v^a} = 0$ owing to the definition of m. Then, if $\overline{\beta v^a} \neq 0$ then the summation $e^{2\pi i \overline{\beta v}} + e^{2\pi i \overline{2\beta v}} + \cdots + e^{2\pi i \overline{m\beta v}}$ is zero. On the other hand, if $\overline{\beta v^a} = 0$ then this summation is m. Thus λ_m^k are

$$\lambda_m^k = m\lambda_k' - m(\overline{\ell'} \bullet \rho + \ell' \bullet \overline{\rho}) \mod 1, \tag{3.23}$$

where the symbol $\bullet$ is defined by $x \bullet y = \sum_{a=1}^3 \delta_{\overline{\alpha v^a},0} \delta_{\overline{\beta v^a},0} x^a y^a$. λ_m^k must be equal to λ_0^k modulo 1 ($\lambda_0^k = 0$). Hence λ_k' are

$$\lambda_k' = \overline{\ell'} \bullet \rho + \ell' \bullet \overline{\rho} + \frac{k}{m}, \tag{3.24}$$

so that λ_n^k satisfy the orthonormal condition (3.20).

In the function $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$, the parameter k is included in the eigenvalue of only $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$. Therefore, in the trace in $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$, the eigenvalues of only $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$ are summed over k,

$$\sum_{k=0}^{m-1} \exp\left\{-2\pi i \left(\overline{\ell'} \bullet \rho + \ell' \bullet \overline{\rho} + \frac{k}{m}\right)\right\} = \begin{cases} \exp\left\{-2\pi i \left(\overline{\ell'} \bullet \rho + \ell' \bullet \overline{\rho}\right)\right\} & (m = 1) \\ 0 & (m > 1) \end{cases}. \tag{3.25}$$

Hence only the states of $m = 1$ contribute to $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. In the states of $m = 1$, $\rho_L^a = \rho_R^a = 0$ for $\forall a$ ($\overline{\alpha v^a} = 0$, $\overline{\beta v^a} \neq 0$) because of the definition of m. If $\rho_L^a = \rho_R^a = 0$ then $\rho^a = \ell^a = 0$ owing to the relations (3.6). Therefore only the states of $\rho^a = \ell^a = 0$ for $\forall a$ ($\overline{\alpha v^a} = 0$, $\overline{\beta v^a} \neq 0$) contributes to $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. Since ρ is the vector in $\overline{\alpha v^a} = 0$ space, only ρ in $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ space

contributes to $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$. I write the momentum in $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ space as ρ_0. In addition, the set of such ℓ as contribute to $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$ is

$$\Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}} = \left\{ \ell \,\middle|\, \ell \in \Lambda,\ \forall a\ (\overline{\alpha v^a} = 0,\ \overline{\beta v^a} \neq 0),\ \ell^a = 0 \right\}. \tag{3.26}$$

In case of $\overline{\alpha v^a} \neq 0$, if $\overline{\beta v^a} = 0$ then $\ell'^a = 0$ because of the fixed point condition (3.15). Therefore the set of such ℓ' as contribute to $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$ is

$$\Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}} = \left\{ \ell' \,\middle|\, \ell' \in \Lambda,\ \forall a\ (\overline{\alpha v^a} \neq 0,\ \overline{\beta v^a} = 0),\ \ell'^a = 0 \right\}. \tag{3.27}$$

But all elements of $\Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}}$ and $\Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}}$ do not necessarily contribute to $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$, because χ satisfied to the both conditions (3.9) and (3.15) must exist. In ref.[5], it is not considered that χ must satisfy the both conditions (3.9) and (3.15). But, if it is not considered then the function $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$ cannot be exactly derived. In this paper, therefore, I introduce the factor

$$\xi^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} = \begin{cases} 1 & \left(\exists\chi,\ \forall a\ (\overline{\alpha v^a} \neq 0,\ \overline{\beta v^a} \neq 0),\ e^{2\pi i \overline{\alpha v^a}}\chi^a + 2\pi\ell^a = e^{2\pi i \overline{\beta v^a}}\chi^a + 2\pi\ell'^a = \chi^a\right) \\ 0 & (\text{otherwise}) \end{cases}. \tag{3.28}$$

In addition, $H_{\rho;\overline{\alpha v},\ell} = \overline{(\rho_0 + \ell/2)} \bullet (\rho_0 + \ell/2)$ and $\tilde{H}_{\rho;\overline{\alpha v},\ell} = \overline{(\rho_0 - \ell/2)} \bullet (\rho_0 - \ell/2)$, because $\overline{\alpha v^a} = 0$, $\overline{\beta v^a} \neq 0$ components of ρ and ℓ are zero, Thus the function (3.17a) becomes

$$\begin{aligned} Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau) = &\sum_{\ell \in \Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}}} \sum_{\ell' \in \Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}}} \xi^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} \\ &\times \int d^d\rho_0 d^d\overline{\rho_0}\, q^{\overline{(\rho_0+\ell/2)}\bullet(\rho_0+\ell/2)} \overline{q}^{\overline{(\rho_0-\ell/2)}\bullet(\rho_0-\ell/2)} e^{-2\pi i(\overline{\ell'}\bullet\rho_0 + \ell'\bullet\overline{\rho_0})}, \end{aligned} \tag{3.29}$$

where d is the dimension of $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ space. The $\overline{\alpha v^a} \neq 0$ or $\overline{\beta v^a} \neq 0$ components of ℓ and ℓ' contribute to only the factor $\xi^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}$ in $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$. Therefore I sum up this factor over ℓ and ℓ' whose $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ components are fixed.

$$B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell'_0} = \sum_{\ell \in \Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}}(\ell_0)} \sum_{\ell' \in \Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}}(\ell'_0)} \xi^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}, \tag{3.30}$$

where ℓ_0 and ℓ'_0 are the vectors of ℓ and ℓ' respectively projected onto $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ space,

$$\Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}}(\ell_0) = \left\{ \ell \,\middle|\, \ell \in \Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}},\ \forall a\ (\overline{\alpha v^a} = \overline{\beta v^a} = 0),\ \ell^a = \ell_0^a \right\}, \tag{3.31a}$$

$$\Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}}(\ell'_0) = \left\{ \ell' \,\middle|\, \ell' \in \Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}},\ \forall a\ (\overline{\alpha v^a} = \overline{\beta v^a} = 0),\ \ell'^a = \ell_0'^a \right\}. \tag{3.31b}$$

$C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}$ must be such constants as $B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell'_0}$ is convergent. Therefore ℓ and ℓ' dependence of this constant is necessary for convergence of $Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau)$. Then the function (3.29) becomes

$$Z_{\rho\frac{\overline{\alpha v}}{\beta v}}(\tau) = \sum_{\ell_0 \in \Lambda_{\ell 0\frac{\overline{\alpha v}}{\beta v}}} \sum_{\ell'_0 \in \Lambda_{\ell' 0\frac{\overline{\alpha v}}{\beta v}}} B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell'_0} \int d^d\rho_0 d^d\overline{\rho_0}\, q^{|\rho_0+\ell_0/2|^2} \overline{q}^{|\rho_0-\ell_0/2|^2} e^{-2\pi i(\overline{\ell'_0}\cdot\rho_0 + \ell'_0\cdot\overline{\rho_0})}, \tag{3.32}$$

where $\Lambda_{\ell 0\frac{\overline{\alpha v}}{\beta v}}$ and $\Lambda_{\ell' 0\frac{\overline{\alpha v}}{\beta v}}$ are the lattices of $\Lambda_{\ell\frac{\overline{\alpha v}}{\beta v}}$ and $\Lambda_{\ell'\frac{\overline{\alpha v}}{\beta v}}$ respectively projected onto $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ space. Since ℓ_0, ℓ'_0 and ρ_0 are the vectors in $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ space, the product of these

is $x_0 \bullet y_0 = x_0 \cdot y_0$. In particular, in case of $\overline{\alpha v^a} \neq 0$ or $\overline{\beta v^a} \neq 0$ for all a, the function (3.32) reads

$$Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) = B^{\overline{\alpha v},0}_{\overline{\beta v},0}, \tag{3.33}$$

because ρ_0, ℓ_0 and ℓ'_0 do not exist. Thus I have derived the function $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$.

In case of the toroidal compactification ($\alpha = \beta = 0$), $\Lambda_{\ell' 0}{}^{0}_{0} = \Lambda$ because of (3.27). Therefore, if B^{0,ℓ_0}_{0,ℓ'_0} do not depend on ℓ'_0 then the summation over ℓ'_0 in the function (3.32) is

$$\sum_{\ell'_0 \in \Lambda} e^{-2\pi i(\overline{\ell'_0} \cdot \rho_0 + \ell'_0 \cdot \overline{\rho_0})} = V_{\Lambda^*} \sum_{\ell''_0 \in \Lambda^*} \delta(\rho_0 - \ell''_0), \tag{3.34}$$

where Λ^* is the dual-lattice of Λ, and V_{Λ^*} is the volume of a unit cell of Λ^*. Hence, only ρ_0 included in Λ^* contribute to the function $Z_{\rho}{}^{0}_{0}(\tau)$. Therefore only the invariant states under $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$ contribute to the partition function, because if $\rho_0 \in \Lambda^*$ then the eigenvalue of $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$ is $e^{-2\pi i(\overline{\ell'_0} \cdot \rho_0 + \ell'_0 \cdot \overline{\rho_0})} = 1$. In the orbifold compactification, however, we must also consider states which is not invariant under $h_{\rho;\overline{\alpha v}}(\overline{\beta v}, \ell')$. The factor $e^{-2\pi i(\overline{\ell'_0} \cdot \rho_0 + \ell'_0 \cdot \overline{\rho_0})}$ is necessary for modular invariance of the partition function (see the next section).

Second, I calculate the function $Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. The function $Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ can be derived by calculating the trace in this function (3.17b) by attention to $N'^a_0 = \tilde{N}^a_0 = 0$,

$$\begin{aligned} Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) = (q\overline{q})^{\overline{\alpha v} \cdot (1 - \overline{\alpha v})/2 - 3/12} \prod_{a=1}^{3} \prod_{n=1}^{\infty} \Bigg\{ & \delta_{\overline{\alpha v^a},0} \left| \left(1 - q^n e^{2\pi i \overline{\beta v^a}}\right)^{-1} \left(1 - q^n e^{-2\pi i \overline{\beta v^a}}\right)^{-1} \right|^2 \\ & + (1 - \delta_{\overline{\alpha v^a},0}) \left| \left(1 - q^{n - \overline{\alpha v^a}} e^{2\pi i \overline{\beta v^a}}\right)^{-1} \left(1 - q^{n + \overline{\alpha v^a} - 1} e^{-2\pi i \overline{\beta v^a}}\right)^{-1} \right|^2 \Bigg\}. \end{aligned} \tag{3.35}$$

Thus I have derived the function $Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$.

I have shown that the partition function $Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ is given by product of the functions (3.32) and (3.35). Finally, I conclude that the complete total partition function $Z(\tau)$ is given by summation of the function $Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$.

4 Modular invariance conditions

In this section, I derive modular invariance conditions of the total partition function $Z(\tau)$. The modular transformation is composed of the following two transformations: the one is $\tau \to \tau + 1$ called T-transformation, the other is $\tau \to -1/\tau$ called S-transformation.

4.1 T-transformation

First, I calculate the T-transformation of the functions $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$, $Z_N{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ and $Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. In case of $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ for some a, the T-transformation of the function (3.32) becomes

$$\begin{aligned} & Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau + 1) \\ & = \sum_{\ell_0 \in \Lambda_{\ell 0}{}^{\overline{\alpha v}}_{\overline{\beta v}}} \sum_{\ell'_0 \in \Lambda_{\ell' 0}{}^{\overline{\alpha v}}_{\overline{\beta v}}} B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell'_0} \int d^d\rho_0 d^d\overline{\rho_0}\, e^{2\pi i(\tau+1)|\rho_0 + \ell_0/2|^2} e^{-2\pi i(\overline{\tau}+1)|\rho_0 - \ell_0/2|^2} e^{-2\pi i(\overline{\ell'_0} \cdot \rho_0 + \ell'_0 \cdot \overline{\rho_0})} \\ & = \sum_{\ell_0 \in \Lambda_{\ell 0}{}^{\overline{\alpha v}}_{\overline{\beta v}}} \sum_{\ell'_0 \in \Lambda_{\ell' 0}{}^{\overline{\alpha v}}_{\overline{\beta v}}} B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell'_0} \int d^d\rho_0 d^d\overline{\rho_0}\, q^{|\rho_0 + \ell_0/2|^2} \overline{q}^{|\rho_0 - \ell_0/2|^2} e^{-2\pi i\{\overline{(\ell'_0 - \ell_0)} \cdot \rho_0 + (\ell'_0 - \ell_0) \cdot \overline{\rho_0}\}}. \end{aligned} \tag{4.1}$$

If $\xi^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} = 1$ then $\ell'^a - \ell^a = 0$ at $\overline{\alpha v^a} \neq 0$, $\overline{\beta v^a - \alpha v^a} = 0$ because of (3.28). Therefore, if $\xi^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'} = 1$ then $\ell' - \ell \in \Lambda_{\ell'}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}$ owing to (3.27). Hence, let ℓ''_0 be $\ell'_0 - \ell_0$, then $\ell''_0 \in \Lambda_{\ell' 0}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}$. In addition, $\Lambda_{\ell 0}{}^{\overline{\alpha v}}_{\overline{\beta v}} = \Lambda_{\ell 0}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}$ because of (3.26). Thus this T-transformation becomes

$$Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau + 1) = \sum_{\ell_0 \in \Lambda_{\ell 0}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}} \sum_{\ell''_0 \in \Lambda_{\ell' 0}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}} B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell''_0 + \ell_0} \int d^d \rho_0 d^d \overline{\rho_0}\, q^{|\rho_0 + \ell_0/2|^2} \overline{q}^{|\rho_0 - \ell_0/2|^2} e^{-2\pi i (\overline{\ell''_0} \cdot \rho_0 + \ell''_0 \cdot \overline{\rho_0})}. \tag{4.2}$$

I assume the condition $B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell''_0 + \ell_0} = F_T B^{\overline{\alpha v},\ell_0}_{\overline{\beta v - \alpha v},\ell''_0}$ for all ℓ_0 and ℓ''_0, where F_T is a constant. Then this T-transformation is $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau + 1) = F_T Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}(\tau)$. This constant is $F_T = B^{\overline{\alpha v},0}_{\overline{\beta v},0} / B^{\overline{\alpha v},0}_{\overline{\beta v - \alpha v},0}$, because if $\ell_0 = \ell''_0 = 0$ then $B^{\overline{\alpha v},0}_{\overline{\beta v},0} = F_T B^{\overline{\alpha v},0}_{\overline{\beta v - \alpha v},0}$. Then this T-transformation is given by

$$Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau + 1) = \frac{B^{\overline{\alpha v},0}_{\overline{\beta v},0}}{B^{\overline{\alpha v},0}_{\overline{\beta v - \alpha v},0}} Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}(\tau). \tag{4.3}$$

The condition $B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell''_0 + \ell_0} = F_T B^{\overline{\alpha v},\ell_0}_{\overline{\beta v - \alpha v},\ell''_0}$ becomes

$$\frac{B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell''_0 + \ell_0}}{B^{\overline{\alpha v},0}_{\overline{\beta v},0}} = \frac{B^{\overline{\alpha v},\ell_0}_{\overline{\beta v - \alpha v},\ell''_0}}{B^{\overline{\alpha v},0}_{\overline{\beta v - \alpha v},0}}. \tag{4.4}$$

In case of $\overline{\alpha v^a} \neq 0$ or $\overline{\beta v^a} \neq 0$ for all a, the function (3.33) are transformed equally as (4.3). Thus I have derived the T-transformation of the function $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$.

The T-transformation of the function (3.35) can be easily calculated,

$$Z_{N}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau + 1) = Z_{N}{}^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}(\tau). \tag{4.5}$$

Thus I have derived the T-transformation of the function $Z_{N}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$.

By using the transformations (4.3) and (4.5), the T-transformation of the partition function $Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ is given by

$$Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau + 1) = \frac{B^{\overline{\alpha v},0}_{\overline{\beta v},0}}{B^{\overline{\alpha v},0}_{\overline{\beta v - \alpha v},0}} Z^{\overline{\alpha v}}_{\overline{\beta v - \alpha v}}(\tau). \tag{4.6}$$

Thus I have derived the T-transformation of the partition function $Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$.

4.2 S-transformation

Second, I calculate the S-transformations of the function $Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$, $Z_{N}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$ and $Z^{\overline{\alpha v}}_{\overline{\beta v}}(\tau)$. In case of $\overline{\alpha v^a} = \overline{\beta v^a} = 0$ for some a, by integrating over ρ_0, the function (3.32) becomes

$$\begin{aligned} Z_{\rho}{}^{\overline{\alpha v}}_{\overline{\beta v}}(\tau) = \frac{1}{(2\tau_2)^d} \sum_{\ell_0 \in \Lambda_{\ell 0}{}^{\overline{\alpha v}}_{\overline{\beta v}}} \sum_{\ell'_0 \in \Lambda_{\ell' 0}{}^{\overline{\alpha v}}_{\overline{\beta v}}} B_{\phi}{}^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell'_0} \\ \times \exp\left\{ -\frac{\pi}{\tau_2} (|\tau|^2 |\ell_0|^2 - \tau_1 \overline{\ell_0} \cdot \ell'_0 - \tau_1 \ell_0 \cdot \overline{\ell'_0} + |\ell'_0|^2) \right\}, \end{aligned} \tag{4.7}$$

where $\tau = \tau_1 + i\tau_2$. The S-transformations of τ_1 and τ_2 are $\tau_1 \to -\tau_1/|\tau|^2$ and $\tau_2 \to \tau_2/|\tau|^2$. Therefore the S-transformation of the function (4.7) becomes

$$\begin{aligned} Z_{\rho\frac{\overline{\alpha v}}{\overline{\beta v}}}(-1/\tau) = \left(\frac{|\tau|^2}{2\tau_2}\right)^d & \sum_{\ell_0 \in \Lambda_{\ell 0}\frac{\overline{\alpha v}}{\overline{\beta v}}} \sum_{\ell_0'' \in \Lambda_{\ell' 0}\frac{\overline{\alpha v}}{\overline{\beta v}}} B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell_0'} \\ & \times \exp\left\{-\frac{\pi}{\tau_2}\left(|\ell_0|^2 + \tau_1\overline{\ell_0}\cdot\ell_0' + \tau_1\ell_0\cdot\overline{\ell_0'} + |\tau|^2|\ell_0'|^2\right)\right\}. \end{aligned} \tag{4.8}$$

I change ℓ_0 and ℓ_0' to $-\ell_0'$ and ℓ_0, respectively. Then $\ell_0 \in \Lambda_{\ell' 0}\frac{\overline{\alpha v}}{\overline{\beta v}} = \Lambda_{\ell 0}\frac{\overline{\beta v}}{-\overline{\alpha v}}$ and $\ell_0' \in \Lambda_{\ell 0}\frac{\overline{\alpha v}}{\overline{\beta v}} = \Lambda_{\ell' 0}\frac{\overline{\beta v}}{-\overline{\alpha v}}$ because of (3.26) and (3.27). Thus this S-transformation becomes

$$\begin{aligned} Z_{\rho\frac{\overline{\alpha v}}{\overline{\beta v}}}(-1/\tau) = \left(\frac{|\tau|^2}{2\tau_2}\right)^d & \sum_{\ell_0 \in \Lambda_{\ell 0}\frac{\overline{\beta v}}{-\overline{\alpha v}}} \sum_{\ell_0' \in \Lambda_{\ell' 0}\frac{\overline{\beta v}}{-\overline{\alpha v}}} B^{\overline{\alpha v},-\ell_0'}_{\overline{\beta v},\ell_0} \\ & \times \exp\left\{-\frac{\pi}{\tau_2}\left(|\tau|^2|\ell_0|^2 - \tau_1\overline{\ell_0}\cdot\ell_0' - \tau_1\ell_0\cdot\overline{\ell_0'} + |\ell_0'|^2\right)\right\}. \end{aligned} \tag{4.9}$$

I assume the condition $B^{\overline{\alpha v},-\ell_0'}_{\overline{\beta v},\ell_0} = F_S B^{\overline{\beta v},\ell_0}_{-\overline{\alpha v},\ell_0'}$ for all ℓ_0 and ℓ_0', where F_S is a constant. Then this S-transformation is $Z_{\rho\frac{\overline{\alpha v}}{\overline{\beta v}}}(-1/\tau) = F_S|\tau|^{2d} Z_{\rho\frac{\overline{\beta v}}{-\overline{\alpha v}}}(\tau)$. This constant is $F_S = B^{\overline{\alpha v},0}_{\overline{\beta v},0}/B^{\overline{\beta v},0}_{-\overline{\alpha v},0}$, because if $\ell_0 = \ell_0' = 0$ then $B^{\overline{\alpha v},0}_{\overline{\beta v},0} = F_S B^{\overline{\beta v},0}_{-\overline{\alpha v},0}$. Then this S-transformation is given by

$$Z_{\rho\frac{\overline{\alpha v}}{\overline{\beta v}}}(-1/\tau) = \frac{B^{\overline{\alpha v},0}_{\overline{\beta v},0}}{B^{\overline{\beta v},0}_{-\overline{\alpha v},0}} |\tau|^{2d} Z_{\rho\frac{\overline{\beta v}}{-\overline{\alpha v}}}(\tau). \tag{4.10}$$

The condition $B^{\overline{\alpha v},-\ell_0'}_{\overline{\beta v},\ell_0} = F_S B^{\overline{\beta v},\ell_0}_{-\overline{\alpha v},\ell_0'}$ becomes

$$\frac{B^{\overline{\alpha v},-\ell_0'}_{\overline{\beta v},\ell_0}}{B^{\overline{\alpha v},0}_{\overline{\beta v},0}} = \frac{B^{\overline{\beta v},\ell_0}_{-\overline{\alpha v},\ell_0'}}{B^{\overline{\beta v},0}_{-\overline{\alpha v},0}}. \tag{4.11}$$

In case of $\overline{\alpha v^a} \neq 0$ or $\overline{\beta v^a} \neq 0$ for all a, the function (3.33) are transformed equally as (4.10) because of $d = 0$. Thus I have derived the S-transformation of the function $Z_{\rho\frac{\overline{\alpha v}}{\overline{\beta v}}}(\tau)$.

Now I rewrite the function (3.35) with the Dedekind eta function $\eta(\tau)$ and the theta function $\vartheta(\nu,\tau)$. The Dedekind eta function and the theta function are defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{4.12a}$$

$$\vartheta(\nu,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+e^{2\pi i\nu}q^{n-1/2})(1+e^{-2\pi i\nu}q^{n-1/2}). \tag{4.12b}$$

Then the function $Z_{N\frac{\overline{\alpha v}}{\overline{\beta v}}}(\tau)$ becomes

$$\begin{aligned} Z_{N\frac{\overline{\alpha v}}{\overline{\beta v}}}(\tau) = \prod_{a=1}^{3}\Bigg\{ & \frac{\delta_{\overline{\alpha v^a},0}\delta_{\overline{\beta v^a},0}}{|[\eta(\tau)]^2|^2} + (q\bar{q})^{-(\overline{\alpha v^a}-1/2)^2/2}\left(\delta_{\overline{\alpha v^a},0}(1-\delta_{\overline{\beta v^a},0})4\sin^2\pi\overline{\beta v^a} + (1-\delta_{\overline{\alpha v^a},0})\right) \\ & \times \left|\frac{\eta(\tau)}{\vartheta((-\overline{\alpha v^a}+1/2)\tau + \overline{\beta v^a}+1/2,\tau)}\right|^2\Bigg\}. \end{aligned} \tag{4.13}$$

The S-transformation of the function (4.13) can be calculated by using the formulas $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$, $\vartheta(-\nu,\tau) = \vartheta(\nu,\tau)$ and $\vartheta(\nu/\tau, -1/\tau) = (-i\tau)^{1/2}e^{\pi i\nu^2/\tau}\vartheta(\nu,\tau)$,

$$
\begin{aligned}
Z_N\frac{\overline{\alpha v}}{\beta v}(-1/\tau) &= \prod_{a=1}^{3}\left\{\frac{\delta_{\overline{\alpha v^a},0}\delta_{\overline{\beta v^a},0}}{|\tau|^2} + \delta_{\overline{\alpha v^a},0}(1-\delta_{\overline{\beta v^a},0})4\sin^2\pi\overline{\beta v^a}\right. \\
&\qquad \left. +\frac{(1-\delta_{\overline{\alpha v^a},0})\delta_{\overline{\beta v^a},0}}{4\sin^2\pi\overline{\alpha v^a}} + (1-\delta_{\overline{\alpha v^a},0})(1-\delta_{\overline{\beta v^a},0})\right\} Z_N\frac{\overline{\beta v}}{-\alpha v}(\tau) \\
&= \frac{F\frac{\overline{\alpha v}}{\beta v}}{F\frac{\overline{\beta v}}{-\alpha v}}\frac{1}{|\tau|^{2d}}Z_N\frac{\overline{\beta v}}{-\alpha v}(\tau),
\end{aligned}
\tag{4.14}
$$

where

$$
F\frac{\overline{\alpha v}}{\beta v} = \prod_{a=1}^{3}\left\{\delta_{\overline{\alpha v^a},0}(1-\delta_{\overline{\beta v^a},0})4\sin^2\pi\overline{\beta v^a} + \delta_{\overline{\alpha v^a},0}\delta_{\overline{\beta v^a},0} + (1-\delta_{\overline{\alpha v^a},0})\right\}. \tag{4.15}
$$

Thus I have derived the S-transformation of the function $Z_N\frac{\overline{\alpha v}}{\beta v}(\tau)$.

By using the transformations (4.10) and (4.14), the S-transformation of the partition function $Z\frac{\overline{\alpha v}}{\beta v}(\tau)$ is given by

$$
Z\frac{\overline{\alpha v}}{\beta v}(-1/\tau) = \frac{B^{\overline{\alpha v},0}_{\overline{\beta v},0}}{B^{\overline{\beta v},0}_{-\alpha v,0}}\frac{F\frac{\overline{\alpha v}}{\beta v}}{F\frac{\overline{\beta v}}{-\alpha v}}Z\frac{\overline{\beta v}}{-\alpha v}(\tau). \tag{4.16}
$$

Thus I have derived the S-transformation of the partition function $Z\frac{\overline{\alpha v}}{\beta v}(\tau)$.

4.3 Modular invariance conditions

Finally, I derive the modular invariance conditions of the total partition function (3.2). By using the T-transformation (4.6), the T-transformation of the total partition function $Z(\tau)$ becomes

$$
Z(\tau+1) = \frac{1}{N}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1}\frac{B^{\overline{\alpha v},0}_{\beta v,0}}{B^{\overline{\alpha v},0}_{\beta v-\alpha v,0}}Z\frac{\overline{\alpha v}}{\beta v-\alpha v}(\tau). \tag{4.17}
$$

I find that the condition $B^{\overline{\alpha v},0}_{\beta v,0} = B^{\overline{\alpha v},0}_{\beta v-\alpha v,0}$ implies T-invariance of $Z(\tau)$: $Z(\tau+1) = Z(\tau)$. By using this condition, the condition (4.4) becomes $B^{\overline{\alpha v},\ell_0}_{\beta v,\ell_0''+\ell_0} = B^{\overline{\alpha v},\ell_0}_{\beta v-\alpha v,\ell_0''}$. In fact, the condition $B^{\overline{\alpha v},\ell_0}_{\beta v,\ell_0''+\ell_0} = B^{\overline{\alpha v},\ell_0}_{\beta v-\alpha v,\ell_0''}$ includes the condition $B^{\overline{\alpha v},0}_{\beta v,0} = B^{\overline{\alpha v},0}_{\beta v-\alpha v,0}$ itself as a special case.

Similarly, by using the S-transformation (4.16), the S-transformation of the total partition function $Z(\tau)$ becomes

$$
Z(-1/\tau) = \frac{1}{N}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1}\frac{B^{\overline{\alpha v},0}_{\beta v,0}}{B^{\overline{\beta v},0}_{-\alpha v,0}}\frac{F\frac{\overline{\alpha v}}{\beta v}}{F\frac{\overline{\beta v}}{-\alpha v}}Z\frac{\overline{\beta v}}{-\alpha v}(\tau). \tag{4.18}
$$

I find that the condition $B^{\overline{\alpha v},0}_{\beta v,0}F\frac{\overline{\alpha v}}{\beta v} = B^{\overline{\beta v},0}_{-\alpha v,0}F\frac{\overline{\beta v}}{-\alpha v}$ implies S-invariance of $Z(\tau)$: $Z(-1/\tau) = Z(\tau)$. By using this condition, the condition (4.11) becomes $B^{\overline{\alpha v},-\ell_0'}_{\beta v,\ell_0}F\frac{\overline{\alpha v}}{\beta v} = B^{\overline{\beta v},\ell_0}_{-\alpha v,\ell_0'}F\frac{\overline{\beta v}}{-\alpha v}$. In fact, the condition $B^{\overline{\alpha v},-\ell_0'}_{\beta v,\ell_0}F\frac{\overline{\alpha v}}{\beta v} = B^{\overline{\beta v},\ell_0}_{-\alpha v,\ell_0'}F\frac{\overline{\beta v}}{-\alpha v}$ includes the condition $B^{\overline{\alpha v},0}_{\beta v,0}F\frac{\overline{\alpha v}}{\beta v} = B^{\overline{\beta v},0}_{-\alpha v,0}F\frac{\overline{\beta v}}{-\alpha v}$ itself as a special case.

Thus I have derived sufficient conditions for the modular invariance of the total partition function are

$$B^{\overline{\alpha v},\ell_0}_{\overline{\beta v},\ell''_0+\ell_0} = B^{\overline{\alpha v},\ell_0}_{\overline{\beta v-\alpha v},\ell''_0}, \quad B^{\overline{\alpha v},-\ell'_0}_{\overline{\beta v},\ell_0} F^{\overline{\alpha v}}_{\overline{\beta v}} = B^{\overline{\beta v},\ell_0}_{\overline{-\alpha v},\ell'_0} F^{\overline{\beta v}}_{\overline{-\alpha v}}. \tag{4.19}$$

5 Summary and remarks

I have constructed the complete 1-loop partition function of a bosonic closed string on orbifolds. In particular, I have paid attention to the following points: the ℓ and ℓ' dependence of the constant $C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}$, the derivation of the eigenstates and eigenvalues of the operator $h_{\rho;\overline{\alpha v}}(\overline{\beta v},\ell')$, and existence of the fixed points which satisfy both conditions (3.9) and (3.15). Furthermore, I have derived sufficient conditions (4.19) for the modular invariance of the total partition function.

In this paper, I particularly discussed the six internal coordinates in the heterotic string. If this argument is adapted to sixteen left-moving bosons (corresponding to sixteen internal coordinates) in the heterotic strings, it is expected to derive higher level current algebras naturally [5]. The higher level current algebras are necessary for grand unified models in the 4-dimensional heterotic string [6]. I think that the ℓ and ℓ' dependence of the constant $C^{\overline{\alpha v},\ell}_{\overline{\beta v},\ell'}$ is necessary for modular invariance of the left-moving bosons on orbifolds.

Acknowledgment

I would like to thank K.I. Kondo for helpful advice.

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